THEOREM ON THE STABILITY OF THE SOLUTION OF A THIRD ORDER DIFFERENTIAL EQUATION WITH A DISCONTINUOUS CHARACTERISTIC

(TEOREMA OB USTOICHIVOSTI RESHENIIA ODNOGO DIFFERENTSIAL'NOGO URAVNENIIA TRET'EGO PORIADKA S RAZRYVNOI KHARAKTERISTIKOI)

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A study is made of the stability of the solutions of a certain third order differential equation which has a discontinuous characteristic. In the general case, the investigated equation describes a definite class of nonlinear feedback control systems. The stability of the solution is attained by increasing a parameter K (the transfer coefficient). It is shown that for a large enough value of K any operating regime of the considered system passes after a certain instant of time into a slipping state. Hereby the dynamic error of the system becomes less than any given number. For the linear case, the given problem has been solved in [1,2].

1. Let us consider the differential equation

$$\dddot{x} + F(x, \dot{x}, \ddot{x}, t) + Kx \operatorname{sign} [x(\ddot{x} - \varphi(x, \dot{x}))] = 0 \qquad (1.1)$$

where K is a positive constant, the function $F(x, \dot{x}, \ddot{x}, t)$ is continuous in all of its arguments in the region

$$|x| < \infty, \qquad |\dot{x}| < \infty, \qquad |\ddot{x}| < \infty, \qquad 0 \leqslant t < \infty$$

is bounded in t for fixed x, $\dot{x}$, $\ddot{x}$, and has continuous first order partial derivatives with respect to x, $\dot{x}$, $\ddot{x}$ and t; the function $\varphi(x, \dot{x})$ is continuous and has piece-wise continuous first and second order partial derivatives with respect to $\dot{x}$ and x in the region

$$|x| < \infty, \qquad |\dot{x}| < \infty$$

Equation (1.1) is equivalent to the system of differential equations

$$\dot{x} = y, \quad \dot{y} = z, \quad \dot{z} = -F(x, y, z, t) - Kx \operatorname{sign}[x(z - \varphi(x, y))] \quad (1.2)$$

Let us impose the following restrictions on the functions $\varphi(x, y)$ and $F(x, y, z, t)$:

a) $|\rho^2 F(x, y/\rho, z/\rho^2, t\rho)| < A(x, y, z), \qquad |\rho\varphi(x, y/\rho)| < B(x,y)$

for sufficiently small values of the parameter ρ; here $A(x, y, z)$ and $B(x, y)$ are assumed to be continuous functions of their arguments

b) $$\begin{array}{ll} \varphi(0, 0) = 0, \varphi(x, 0)x < 0 & \text{for } x \neq 0, \\ [\varphi(x,y) - \varphi(x, 0)]y < 0 & \text{for } y \neq 0, \end{array} \qquad \int_{\pm\infty}^{0} \varphi(x, 0)\, dx = \infty$$

We note that the condition (a) is satisfied when the function $F(x, y, z, t)$ is linear in x, y, z, and is bounded in t for $0 \leqslant t < \infty$. Any linear function $\varphi(x, y) = cx + dy$, where c and d are constants, also satisfies condition (a); furthermore, it will also satisfy condition (b) if $c < 0$ and $d < 0$.

In the general case equation (1.1) describes a control system of a nonlinear object which reacts to an arbitrary input. For the linear case this equation was considered in [1,2], where a study was made of a third order control system of a variable structure. The theory of similar systems has also been developed in [3,4]. We shall now formulate the basic results of our work.

Theorem. Let conditions (a) and (b) be satisfied, and let ε be a given positive number. Then, for the given bounded region G of the phase space, there exists a positive number K_0 such that for every $K \geqslant K_0$, any solution of system (1.2) whose initial value lies in G will satisfy after some instant of time the condition

$$|x(t)| < \varepsilon, \quad |y(t)| < \varepsilon, \quad |z(t)| < \varepsilon$$

Note 1. Certain problems on optimal control lead to systems of type (1.2). Indeed, let us consider the system of differential equations

$$\dot{x} = y, \qquad \dot{y} = z, \qquad \dot{z} = -F(x, y, z, t) - Ku(t)x \qquad (1.3)$$

and let us attempt to find a function $u(t)$ such that, for $|u(t)| \leqslant 1$, some function $v(x, y, z)$ will have, because of system (1.3), the maximum velocity of decrease [5]. It is not difficult to see that the function $u(t)$ must, in this case, have the form $u(t) = \text{sign}(x\partial v/\partial x)$. But this leads, under some additional hypotheses, to system (1.2). If one looks for a function $u(t)$ which will minimize some functional, then, making use of the maximum principle of L.S. Pontriagin, one again is led to a system of type (1.2).

Note 2. In the general case, system (1.2) is a system of differential equations in which the right side of the third equation has a discontinuity on the surface $z = \varphi(x, y)$. The theory of systems with discontinuous right-hand sides has been well developed in [6-8].

The plane $x = 0$ is a plane of switching in the sense that the quantity $n = \text{sign}[x(z - \varphi(x, y))]$, which appears in an equation of system (1.2), changes sign. However, this plane, as can be easily seen, is not a plane of discontinuity; the trajectories of system (1.2) make a "suture" with the plane $x = 0$. A different situation occurs on the surface (S) given by equation $z = \varphi(x, y)$.

Let us introduce the notation $R = z - \varphi(x, y)$. Then one can find at every point $M(x, y, \varphi(x, y))$ of the surface (S), at a fixed instant of time, the quantities

$$N_1 = \lim_{R \to +0} dR / dt, \qquad N_2 = \lim_{R \to -0} dR / dt$$

where dR/dt can be computed by means of system (1.2). If hereby we find that $N_1 N_2 > 0$, then the given point of the surface (S) will be a point of a suture, that is, the trajectory of system (1.2) will intersect the surface (S) at the point M, and will pass from one part of the phase space relative to the surface (S) into another part. If however, $N_1 < 0$ and $N_2 > 0$, we shall have a more interesting case of the position of the trajectories of system (1.2), the case of "slipping". In this case the vectors of the system which act on both sides of the surface (S) will be directed to the surface itself. Hence, the point, of the phase space, having fallen on the surface (S) must remain on it until the above indicated inequalities for N_1 and N_2 are fulfilled. In order to find the vector of the velocity of slipping of the point on the surface (S), one usually makes use of the following rule [6]. From the given point of the surface, one extends velocity vectors which are determined by systems acting from one as well as from the other side of the surface. The ends of the constructed vectors are connected by a straight line. The point of intersection of this straight line with the tangent plane, drawn to the surface at the given point, is taken as the end of the vector of slipping of the point on the surface (S). Starting out with this rule, and ascribing to the point $(x, y, \varphi(x, y))$ of the surface (S) the coordinates (x, y), one can easily obtain, in the regions of slipping of this surface, the system of equations

$$\dot{x} = y, \qquad \dot{y} = \varphi(x, y) \tag{1.4}$$

which describes the regime of slipping.

We note that this regime of slipping which is described by system (1.4) does not depend on the form of the function $F(x, y, z, t)$. This observation is important if one takes into consideration the fact that by increasing the value of K one can make any point (other than the origin) of the surface (S) be a point of slipping of the system (1.2).

If, in some way, we could make the points of the phase space of the system fall on the surface (S), and then (while in the regime of slipping) make them move in the neighborhood of the origin, then we would obtain, in some sense, a property of stability of the solution of system (1.2). Hereby, this property of stability would appear as a rough one relative to the variation of the function $F(x, y, z, t)$ which does not disturb the existence of the slipping regime.

2. *Proof of the theorem.* We shall first show that any point of the region G (the region of possible initial positions), which moves in accordance with the equations of system (1.2), will at some time reach the surface (S). For this we introduce a change of variables in the system

$$t = \rho\tau, \qquad X = x, \qquad Y = \rho y, \qquad Z = \rho^2 z \qquad (\rho = K^{-1/3}) \tag{2.1}$$

The new system can be written in the form

$$\frac{dX}{d\tau} = Y, \qquad \frac{dY}{d\tau} = Z, \qquad \frac{dZ}{d\tau} = -nX - \rho^3 F(X, Y/\rho, Z/\rho^2, \rho\tau)] \tag{2.2}$$
$$n = \operatorname{sign}[X(Z - \rho^2\varphi(X, Y/\rho)]$$

The quantity ρ plays the role of a small parameter.

Along with system (2.2), which is equivalent to system (1.2), we will consider also the simplified system

$$\frac{dX}{d\tau} = Y, \qquad \frac{dY}{d\tau} = Z, \qquad \frac{dZ}{d\tau} = -nX \tag{2.3}$$

We shall first show that a point $M(t)$ that moves along a trajectory of this system will reach the surface (S). Let us denote by G_ρ the image of the region G under the transformation (2.1).

We now introduce the notation $Q = Z - \rho^2\varphi(X, Y/\rho)$ and divide the region G_ρ into four sub-regions G_1, G_2, G_3 and G_4, by defining them in the following way:

$$Q > 0, \quad X > 0 \quad (G_1), \qquad Q < 0, \quad X < 0 \quad (G_3)$$
$$Q > 0, \quad X < 0 \quad (G_2), \qquad Q < 0, \quad X > 0 \quad (G_4)$$

Let us assume that the initial point $M_0(X_0, Y_0, Z_0)$ of a trajectory lies in the region G_2. It is easy to see that the solution of system (2.3) in the region G_2 can be written in the form

$$X = ce^\tau + \varphi_1(\tau), \qquad Y = ce^\tau + \varphi_2(\tau), \qquad Z = ce^\tau + \varphi_3(\tau)$$
$$c = {}^1/_3\,(X_0 + Y_0 + Z_0)$$

The functions $\varphi_1(\tau)$, $\varphi_2(\tau)$ and $\varphi_3(\tau)$ tend towards zero when $\tau \to \infty$. If the coordinates of the point M_0 satisfy the conditions $X_0 + Y_0 + Z_0 > 0$, then the negative abscissa of the point $M(\tau)$ will necessarily change sign when τ increases. Therefore, if the point $M(\tau)$ does not reach the surface (S), then it must reach the plane $X = 0$, and at this instant $Y > 0$. If, however, $X_0 + Y_0 + Z_0 < 0$, then we shall have the next equation for the trajectory which starts out from the point M_0

$$Q(\tau) = {}^1/_3\,(X_0 + Y_0 + Z_0)\,e^\tau + \varphi_3(\tau) - \rho^2\varphi\,(X(\tau), Y(\tau)\,/\,\rho)$$

Suppose that a point $M(\tau)$ starts in the region G_2. Then its abscissa satisfies the inequality $X(\tau) < 0$ for all τ; we shall also have $Y(\tau) < 0$ as y increases. But condition (b) implies the validity of inequality $\varphi(X(\tau), Y(\tau)/\rho > 0$. This means that the quantity $Q(\tau)$, which is positive in the region G_2, must become zero when τ increases, i.e. the image point of system (2.3) must reach the surface (S) as τ increases. If, however, one assumes that under the condition that $X_0 + Y_0 + Z_0 < 0$ for the initial point M_0 of the trajectory, the image point $M(\tau)$ leaves the region G_2, then it is at once clear that the point $M(\tau)$ can reach the plane $X = 0$ only between the lines $Y + Z = 0$ and $Z = \varphi(0, Y/\rho)$. From condition (b) it follows that the line $Z = \varphi(0, Y/\rho)$, with $Z > 0$, is located in the region $Y < 0$.

Therefore, if the point $M(\tau)$ starts out from the region G_2, then $Y < 0$ at the point where $M(\tau)$ falls on the plane $X = 0$. This leads to a contradiction, because the inequality $dX/d\tau = Y < 0$ implies the impossibility of an increase in X at the time of intersection of the point $M(\tau)$ with the plane $X = 0$ while the point $M(\tau)$ is passing from the region G_2, where $X < 0$, into the region G_1, where $X > 0$.

And thus, since the initial point M_0 of the trajectory lies in the region G_2, for any sufficiently small value of ρ, the image point $M(\tau)$

of system (2.3) either falls on the surface (S) when t increases, or it passes into the region G_1, thereby intersecting the plane $X = 0$ in that part where $Y > 0$.

Suppose now that the initial point M_0 lies in the region G_1. As long as the image point $M(\tau)$ of system (2.3) does not leave the region G_1, we have

$$X(\tau) = c_1 e^{-\tau} + e^{1/2\tau}\left[c_2 \cos\left(1/2\sqrt{3}\,\tau\right) + c_3 \sin\left(1/2\sqrt{3}\,\tau\right)\right] \qquad (2.4)$$

Here c_1, c_2 and c_3 are some constants, where $c_1 = (X_0 - Y_0 + Z_0)/3$.

Two possibilities can occur: either the point $M(\tau)$ reaches, in a finite time τ, the surface (S), or its abscissa $X(\tau)$ changes its sign as the point moves, i.e. the point $M(\tau)$ meets the plane $X = 0$. The only exceptions are the points which lie on the integral straight line $X = -Y = Z$; they approach the origin asymptotically with an increase in τ.

The points which pass from the region G_1 into the region G_2 through the plane $X = 0$ either meet there the surface (S) or they return again into the region G_1 through the half-plane $X = 0$, $Y > 0$. We shall show that after its return to the region G_1 the point $M(\tau)$ meets the plane (S) without leaving G_1. For this purpose we consider one of such points with the coordinates $X_0 = 0$, $Y_0 > 0$, $Z_0 \geqslant 0$, and we assume that the function $X(\tau)$ given by relation (2.4) becomes zero for the first time when $\tau = \tau_1$. By Rolle's theorem, the function $Y = dX/d\tau$ will vanish on the interval $(0, \tau_1)$ at least once; let this happen the first time at $\tau = \tau_2 \leqslant \tau_1$. Since $Z(0) = Z_0 \geqslant 0$, the function $Y(t)$ will increase at the point $\tau = 0$. Therefore, we can apply again Rolle's theorem. Thus we find a number τ_3 such that $0 < \tau_3 \leqslant \tau_2$ and $Z(\tau_3) = 0$. But this means that on the interval $(0, \tau_1)$ there is a point at which the function $Z(\tau)$ takes on a negative value. Hence, the quantity $Q = Z - \rho^2\varphi(X, Y/\rho)$ must vanish, for sufficiently small ρ at least once before the point $M(\tau)$ falls again on the plane $X = 0$.

Finally, let us consider an initial point M_0 from the region G_1, the coordinates of which satisfy the conditions $X_0 = 0$, $Y_0 > 0$, $Z_0 < 0$. It is easy to see that such a point is located between the integral plane $X - Y + Z = 0$ of system (2.3) with $n = 1$ and the surface (S). Since these surfaces intersect the plane $X = 0$ in the lines $Z = Y$ and $Z = \rho^2\varphi(0, Y/\rho)$, it is obvious that the point $M(\tau)$ cannot enter into the region $X = 0$, $Y < 0$ without having first met the surface (S).

Thus, if the initial point M_0 belongs to the region G_1, then, for sufficiently small ρ, the image point $M(\tau)$ of system (2.3) will either at once meet the surface (S) as τ increases, or it will pass into the region G_2 and there meet the surface (S), or it will pass into the region G_2 and from there return to the region G_1 and there meet (S).

Entirely analogous considerations can be made for initial points that lie in the regions G_3 and G_4.

It has thus been proved that for small enough ρ, the point $M(\tau)$ which lies on $G\rho$ and moves in accordance with the equations of the simplified system (2.3), will either meet the surface (S) in a finite time τ, or it will approach the origin of the coordinate system asymptotically when the point $M(\tau)$ belongs to the straight line $X = -Y = Z$. In view of the theorem on the continuous dependence of the solution on the parameter, one can make analogous conclusions for system (2.2) and hence also for system (1.2). In other words, we have proved that for a sufficiently large value of K, the point $M(t)$ which lies in the region G of possible initial conditions and which moves according to system (1.2), will meet the surface of switching (S) when t increases.

We now pass to the second part of the proof of the theorem; we shall study the behavior of the image of the point $M(t)$ which under the action of system (1.2) reaches the surface of switching (S). For this purpose we consider a new coordinate system in which x and y remain the same, but the third coordinate is $R = z - \varphi(x, y)$. Then system (1.2) is transformed into an equivalent system of differential equations

$$\dot{x} = y, \qquad \dot{y} = R + \varphi(x, y), \qquad \dot{R} = -\varphi_y'(x, y)R - $$
$$- F(x, y, R + \varphi(x, y), t) - Knx - \varphi_x'(x, y)y - \varphi_y'(x, y)\varphi(x, y)$$
$$(n = \operatorname{sign} xR) \tag{2.5}$$

Under such a coordinate transformation the surface of the change-over $z = \varphi(x, y)$ passes into the plane of switching $R = 0$. Hence, by examining the behavior of the image point which under the action of system (2.5) reaches the plane $R = 0$, we can draw corresponding conclusions about the nature of the behavior of the image point of system (1.2) with respect to the surface (S).

On the plane $R = 0$, the quantity $\dot{R}$ takes the form

$$\dot{R} = -Knx - \Phi(x, y, t) \tag{2.6}$$

where we have used the notation

$$\Phi(x, y, t) = F[x, y, \varphi(x, y), t) + \varphi_x'(x, y)y + \varphi_y'(x, y)\varphi(x, y) \tag{2.7}$$

Let us denote by D the region of the plane $R = 0$ which consists of the ends of the arcs of the trajectories which start in the region G. From the preceding part of the proof it follows that the region D is a uniformly bounded region when $K \to \infty$. Indeed, the transformation (2.1) transforms the region G into the region G_ρ, whereby it is easily seen that if $\rho_1 > \rho_2$ we shall have $G_{\rho_1} > G_{\rho_2}$. The points of G_{ρ_1} and G_{ρ_2}, moving along the trajectories of system (2.3) pass into points of the regions D_{ρ_1} and D_{ρ_2}, which lie in the plane $R = 0$, whereby $D_{\rho_1} \supset D_{\rho_2}$. By the theorem on the continuous dependence of the solutions on the parameter, we can draw a conclusion on the boundedness of the region which consists of the points of the surface (S) that have come out of the region G_ρ along trajectories of system (2.2). Returning then to the old coordinates, we obtain the boundedness of the region D when $\rho \to \infty$.

On the basis of the above formulated hypotheses relative to the functions $F(x, y, z, t)$ and $\varphi(x, y)$, we have the following relation in the region D:

$$|\Phi(x, y, t)| < m \tag{2.8}$$

From equation (2.6) one can easily see that the straight lines $x = \Delta x$ and $x = -\Delta x$, where $\Delta x = m/K$, separate (in the plane $R = 0$) a strip $|x| \leqslant \Delta x$ outside of which the sign of the derivative $\dot{R}$ for system (2.5) is given by the relation sign $\dot{R} = -n$ sign x. An analysis of this formula shows that the points of the plane $R = 0$, which are located outside the indicated strip, are points of slipping of system (2.5). The motion of the image point on this part of the plane $R = 0$ (outside the strip $|x| \leqslant \Delta x$) are defined by the limiting differential equation $\ddot{x} = \varphi(x, \dot{x})$ which is equivalent to the system of equations (1.4).

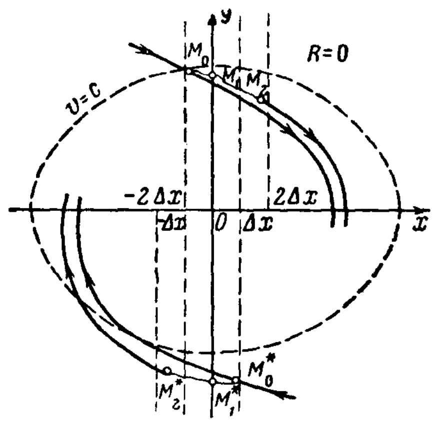
Inside the strip $|x| \leqslant \Delta x$ of the plane $R = 0$ the sign of the derivative $\dot{R}$ is undetermined. The strip contains the regions of "sutures" bounded by the curves

$$\Phi(x, y, t) + Knx = 0 \tag{2.9}$$

where the trajectories of system (2.5) cut the plane $R = 0$ with an increase of time.

Thus, the image point $M(t)$ of system (2.5) after reaching the plane $R = 0$, moves in it in accordance with system (1.4) until it comes at the time $t = t_0 > 0$ to the boundary of the region of "sutures", i.e. to one of the curves (2.9) at some point $M_0(x_0, y_0)$. The actual fact of reaching this point is caused by the stability in the large of the zero solution of system (1.4). This stability follows from the condition (b) for the function $\varphi(x, y)$ [9].

Later, as the time t increases, the image point $M(t)$ will meet the plane $R = 0$ and it will begin to move under the action of the equations of system (2.5) either in the half-space $R > 0$ (when $n = -1$), or in the half-space $R < 0$ (when $n = 1$). We shall show that for large enough values of K the point $M(t)$ of system (2.5), after having remained a sufficiently short time outside the plane $R = 0$, returns to this plane at a point which lies sufficiently close to y.



Let us suppose for the sake of definiteness that the point M_0 is located in the second quadrant of the plane $R = 0$, i.e. its coordinates satisfy the conditions $-\Delta x \leqslant x_0 < 0$, $y_0 > 0$ (see figure). Further, let us assume that the following relation holds

$$y_0 \gg \Delta x \tag{2.10}$$

Making use of the equations

$$y\, dy \,/\, dx = R + \varphi(x, y)$$
$$y\, dR \,/\, dx = -\varphi_y' R - F(x, y, R + \varphi(x, y), t) - Knx - \varphi_x' y - \varphi_y' \varphi \tag{2.11}$$

which were obtained from the differential equations of system (2.5), we find an approximate solution $y(x)$ and $R(x)$ of system (2.11) which corresponds to the trajectory of the image point $M(t)$ with the initial point $M_0(x_0, y_0) = M(t_0)$. Since for the point M_0, $y_0 > 0$ by hypothesis, it follows that with an increase in time $t > t_0$, the image point, after having reached the plane $R = 0$, will move in the direction of increasing x, i.e. in the direction towards the plane $x = 0$. On the other hand, since $|x_0| \leqslant \Delta x$, and the quantity $\Delta x = m/K$ can be made as small as we please by increasing K, we shall seek the solution in the interval $[x_0, 0]$, and for the finding of it we shall restrict ourselves to its first approximation

$$y(x) = y_0 + \varphi(x_0, y_0) \,/\, y_0 (x - x_0), \qquad R(x) = -Knx_0 - \Phi(x_0, y_0, t_0) \,/\, y_0 (x - x_0)$$

The obtained formulas make it possible to find the approximate value of the coordinates of the point $M_1(x_1, y_1, R_1)$, the point at which the trajectory cuts the plane $x = 0$ for some $t = t_1 > t_0$. Indeed, we obtain the equations

$$x_1 = 0, \qquad y_1 = y_0 - \varphi(x_0, y_0)\, x_0 \,/\, y_0 \approx y_0, \qquad R_1 \approx 0$$

which are valid for sufficiently large values of K when one may neglect terms of the same order of magnitudes as Δx.

From the point M_1, the point $M(t)$ moves, as the time $t > t_1$ increases, into the half-space $x > 0$ as long as the ordinate of the point remains positive. Since the ordinate of the point M_1 is again positive, in view of relation (2.10), the image point $M(t)$ will move as time increases from the point M_1 in the direction of increase of $x(t)$ at least for positive x which are sufficiently small compared to the number Δx. Let us find the solution $R(x)$ of system (2.11), which corresponds to the trajectory that emanates from the point M_1. We want this solution in the form of a power series in x, and we shall neglect the terms of degree higher than two.

Hereby we shall take into account the boundedness of the functions $F(x, y, z, t)$, $\varphi(x, y)$, and their derivatives, and we shall also assume that K is sufficiently large.

Then we shall have

$$R(x) = -\Phi(0, y_0, t_1)\, x / y_0 - Kn\, x^2/2y_0 \tag{2.12}$$

from which we find the value of x for which the function $R(x)$ becomes zero. At the same time we find the value of the abscissa of the point $M_2 = M(t_2)$, the point at which the image point returns to the plane $R = 0$ when $t = t_2 > t_1$.

For the abscissa of the point M_2 we obviously obtain

$$x_2 = -2\Phi(0, y_0, t_1) / Kn$$

Taking into account condition (2.8) we obtain the relation $|x_2| \leqslant 2\Delta x$. For the trajectory under consideration we obviously have $x_2 > 0$, since the value of the ordinate of the point M_2 differs from the value of the ordinate of the initial point M_0 of the trajectory by a quantity whose order of magnitude is that of Δx; i.e. on the interval $|-\Delta x, 2\Delta x|$, the sign of the ordinate of the point $M(t)$ will remain positive in our case.

Thus we have shown that for a sufficiently large value of K the image point $M(t)$ of system (2.5), after having met the plane $R = 0$ at the point $M_0(x_0, y_0)$ the coordinates of which are such that $|x_0| \leqslant \Delta x$, $y_0 \gg \Delta x$, will again return to the plane $R = 0$ at a point $M_2(x_2, y_2)$ for which the ordinate is again positive and comparable to the number y_0, while the abscissa does not exceed the number $2\Delta x$ (Fig.). The difference between the abscissas M_0 and M_2 is not greater than $3\Delta x$. Therefore, the time of passage of the image point from the point M_0 to the point M_2 will be short if K is sufficiently large.

Analogous arguments can be made for the trajectories of system (2.5), which meet the plane $R = 0$ at some point $M_0^*(x_0^*, y_0^*)$ where $y_0^* < 0$ and $0 < x_0^* \leqslant \Delta x$ if the condition $|y_0^*| \gg \Delta x$ is fulfilled. In other words, in this case it can be shown that the image point $M(t)$, having left the plane $R = 0$ at the point M_0^*, will cut the plane $x = 0$, and then having remained a short time outside this plane, will again return to this plane in the region of slippage at the point $M_2^*(x_2^*, y_2^*)$, the coordinates of which (see figure) satisfy the relation $x_2^* < 0$, $|x_2^*| \leqslant 2\Delta x$. Thus the presence of the strip $|x| \leqslant \Delta x$ which contains the regions of sutures of system (2.5) causes a deflection of the considered trajectories from the plane $R = 0$. This deflection occurs on a strip which does not leave the strip $|x| \leqslant 2\Delta x$. Since $\Delta x = m/K$, where m is the constant from relation (2.8), and K is the parameter of the system (2.5), it follows that one can always select a value of K so large that Δx can be made as small as we please. Simultaneously with the decrease of Δx, the maximum deflection of the function $R(x)$ from zero, as well as the time during which the image point of the considered trajectory remains outside the plane $R = 0$, become arbitrarily small.

In the first part of the proof of the theorem it was established that from any point of phase space the image point of system (1.2) will reach, with an increase in time, the surface (S), where $R = 0$. The further behavior of the image point is determined by its motion on this surface and its deflection (or deviation) from this surface which is determined by the existence of regions of "sutures" of the system.

Let us now consider, on the plane $R = 0$, in addition to system (1.4), also the system

$$\dot{x} = y, \qquad \dot{y} = \varphi(x, y) + R(t) \tag{2.13}$$

which is formed with the first two equations of system (2.5). By $R(t)$ we denote here the value of the quantity R in the process of the motion of the point $M(t)$ studied above. Therefore $R(t) = 0$ if the point slides along the plane $R = 0$; $R(t) \neq 0$ if the point $M(t)$ is not on this plane. Therefore, system (2.13) describes the motion of the projection of the point $M(t)$ on the plane $R = 0$.

It has been shown above that for large enough K the function $R(t)$ can be made arbitrarily small in absolute value. Further, because of the fulfillment of the condition (b), the zero position of equilibrium is for system (1.4) asymptotically stable for arbitrary initial disturbances [9]. Let us now consider the function

$$v = y^2 - 2\int_0^x \varphi(x, 0)\, dx$$

The derivative of this function, taken with the consideration of (2.13), has the form

$$\dot{v} = 2y\,[\varphi(x, y) - \varphi(x, 0)] - 2R(t)\,y = 2y\varphi(0, y) + \\ + 2y\,[\varphi(x, y) - \varphi(0, y)] - 2y\varphi(x, 0) - 2y\,R(t)$$

From (2.12) it can be easily deduced that $|yR(t)| < m^2/2K$, where m is a positive constant. On the other hand, if y_0 is an arbitrary small number then, taking into account condition (b), we can deduce the inequality $y\varphi(0, y) < 0$, and also that the ordinate y of any point of the region D is bounded, and that, if $|y| > y_0$, we have $y\varphi(0, y) < -r_0$, where r_0 is some positive constant.

Next, selecting the value of K large enough, we can obtain, in view of the continuity of the function $\varphi(x, y)$, for points of the strip $|x| \leqslant 2\Delta x$ (within the region D) the following inequality

$$|y\,[\varphi(x, y) - \varphi(0, y)]| < {}^1/_4\, r_0, \qquad |y\varphi(x, 0)| < {}^1/_4 r_0$$

Finally, if $K > 2m^2/r_0$, we shall have also the relation $|yR(t)| < r_0/4$. Thus in the region $|x| \leqslant 2\Delta x$ with $|y| > y_0$ we have for the derivative $\dot{v}$ the inequality $\dot{v} < -r_0/2 < 0$. Furthermore, $\dot{v} < 0$ everywhere in the region D outside the strip $|x| \leqslant 2\Delta x$. From this it follows that by choosing K sufficiently large we can make sure that the derivative $\dot{v}$ will remain negative everywhere in the region D, except for an arbitrarily small neighborhood of the origin of the coordinate system. But from the presented arguments there follows, in an obvious manner, the validity of our theorem.

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